

ON SOME GEOMETRIC PROPERTIES OF QUASI-PRODUCT PRODUCTION MODELS

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ABSTRACT. In this paper we obtain classification results on the quasi-product production functions in terms of the geometry of their associated graph hypersurfaces. In particular, we give a complete classification of quasi-product production models whose production hypersurfaces have null Gauss-Kronecker curvature, generalizing in a new setting some recent results concerning basic production models. We also classify quasi-product production functions with constant elasticity of production with respect to any factor of production, with proportional marginal rate of substitution and with constant elasticity of substitution property. In particular, we obtain several results on the geometry of Spillman-Mitscherlich and transcendental production functions.

Keywords: quasi-product production function, production hypersurface, mean curvature, Gauss-Kronecker curvature, sectional curvature, constant return to scale, marginal rate of substitution, constant elasticity of substitution.

2010 Mathematics Subject Classification: 53A07, 91B02, 91B15.

1. INTRODUCTION

The notion of *production function* is a key concept in economics, being used in the mathematical modeling of the relationship between the output of a firm, an industry, or an entire economy, and the inputs that have been used in obtaining it. Generally, a production function is a twice differentiable mapping f from a domain D of $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0, \dots, x_n > 0\}$ into $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$, where \mathbb{R} denotes the set of real numbers. Hence we have $f : D \subset \mathbb{R}_+^n \rightarrow \mathbb{R}_+$, $f = f(x_1, \dots, x_n)$, where f is the quantity of output, n is the number of the inputs and x_1, \dots, x_n are the factor inputs, such as: labor, capital, land, raw materials etc. We note that some historical information about the evolution of the concept of production functions and a lot of interesting examples can be found in [20]. We only recall that, among the family of production functions, the most famous is the Cobb-Douglas production function introduced in [14] in order to describe the distribution of the national income of the USA. A generalized Cobb-Douglas production function depending on n -inputs ($n \geq 2$) is given by [26]

$$(1) \quad f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^n x_i^{k_i},$$

where $A > 0$, $k_1, \dots, k_n \neq 0$. It is obvious that a generalized Cobb-Douglas production function is homogeneous of degree $p = \sum_{i=1}^n k_i$. We recall that the homogeneity has a precise economic interpretation: if the inputs are multiplied by same factor,

then the output is multiplied by some power of this factor. If $p = 1$ then the function is said to have a *constant return to scale*, if $p > 1$ then we have an *increased return to scale* and if $p < 1$ then we say that the function has a *decreased return to scale*. We note that Cobb-Douglas production functions were generalized by H. Uzawa [21] and D. McFadden [18] by introducing a new production function, usually called generalized ACMS production function, Armington aggregator or generalized CES production function, by

$$(2) \quad f(x_1, \dots, x_n) = A \left(\sum_{i=1}^n k_i x_i^\rho \right)^{\frac{\gamma}{\rho}}, \quad (x_1, \dots, x_n) \in D \subset \mathbb{R}_+^n,$$

with $A, k_1, \dots, k_n, \rho \neq 0$, where γ is the degree of homogeneity.

It is well known that the classical treatment of the production functions makes use of the projections of production functions on a plane, but, unfortunately, this approach leads to limited conclusions and a differential geometric treatment is more than useful. We note that this approach is feasible since any production function f can be identified with the graph of f , *i.e.* the nonparametric hypersurface of the $(n + 1)$ -dimensional Euclidean space \mathbb{E}^{n+1} defined by

$$(3) \quad L(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n))$$

and called the *production hypersurface* of f [23]. Using this treatment, a surprising link between some basic concepts in the theory of production functions and the differential geometry of hypersurfaces was obtained in [22]: a generalized Cobb-Douglas production function has decreasing/increasing return to scale if and only if the corresponding hypersurface has positive/negative Gauss-Kronecker curvature. Moreover, this production function has constant return to scale if and only if the corresponding hypersurface has vanishing Gauss-Kronecker curvature. Moreover, in [13], the authors proved that a homogeneous production function with an arbitrary number of inputs defines a flat hypersurface if and only if either it has constant return to scale or it is a multinomial production function. This result was generalized by X. Wang to the case of homogeneous hypersurfaces with constant sectional curvature [25]. On the other hand, other classes of production functions, like quasi-sum production functions and homothetic production functions, were investigated via geometric properties of their associated graph hypersurfaces in Euclidean spaces (see, e.g., [5, 9, 12] and references therein). We outline that such kind of results are of great interest not only in economic analysis [1, 19], but also in the classical differential geometry, where the study of hypersurfaces with certain curvature properties is one of the basic problems [6].

Motivated by the above works, in the present paper we derive the main properties of quasi-product production models in economics in terms of the geometry of their graph hypersurfaces. In particular, we give a classification of quasi-product production functions whose production hypersurfaces have null Gauss-Kronecker curvature, generalizing in a new setting some recent results concerning quasi-sum and homothetic production models [9, 12]. Moreover, we classify quasi-product production functions with constant elasticity of production with respect to any factor of production and with proportional marginal rate of substitution, extending in a new setting some recent results concerning quasi-sum production functions [24].

2. PRELIMINARIES ON THE GEOMETRY OF HYPERSURFACES

For general references on the geometry of hypersurfaces, we refer to [6, 7].

If M is a hypersurface of the Euclidean space \mathbb{E}^{n+1} , then it is known that the *Gauss map* $\nu : M \rightarrow S^n$ maps M to the unit hypersphere S^n of \mathbb{E}^{n+1} . With the help of the differential $d\nu$ of ν it can be defined a linear operator on the tangent space $T_p M$, denoted by S_p and known as the *shape operator*, by

$$g(S_p v, w) = g(d\nu(v), w)$$

for $v, w \in T_p M$, where g is the metric tensor on M induced from the Euclidean metric on \mathbb{E}^{n+1} .

The eigenvalues of the shape operator are called *principal curvatures*. The determinant of the shape operator S_p , denoted by $K(p)$, is called the *Gauss-Kronecker curvature*. When $n = 2$, the Gauss-Kronecker curvature is simply called the *Gauss curvature*. We recall that a *developable surface* is a surface having null Gauss curvature. The trace of the shape operator S_p is called the *mean curvature* of the hypersurfaces. A hypersurface is said to be *minimal* if its mean curvature vanishes identically.

The Riemann curvature tensor R of M is given by

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w,$$

where ∇ is the Levi-Civita connection of g . A Riemannian manifold is said to be *flat* if its Riemann curvature tensor vanishes identically.

We denote the partial derivatives $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, ..., etc. by f_{x_i} , $f_{x_i x_j}$, ..., etc. We also put

$$w = \sqrt{1 + \sum_{i=1}^n f_{x_i}^2}.$$

Next we recall the following well-known result for later use.

Lemma 2.1. [9] *For the production hypersurface of \mathbb{E}^{n+1} defined by (3), one has the following.*

i. *The Gauss-Kronecker curvature K is given by*

$$(4) \quad K = \frac{\det(f_{x_i x_j})}{w^{n+2}}.$$

ii. *The mean curvature H is given by*

$$(5) \quad H = \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{f_{x_i}}{w} \right).$$

iii. *The sectional curvature K_{ij} of the plane section spanned by $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial x_j}$ is*

$$(6) \quad K_{ij} = \frac{f_{x_i x_i} f_{x_j x_j} - f_{x_i x_j}^2}{w^2 (1 + f_{x_i}^2 + f_{x_j}^2)}.$$

iv. *The Riemann curvature tensor R and the metric tensor g satisfy*

$$(7) \quad g \left(R \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell} \right) = \frac{f_{x_i x_\ell} f_{x_j x_k} - f_{x_i x_k} f_{x_j x_\ell}}{w^4}.$$

3. QUASI PRODUCT PRODUCTION MODELS

There are some special classes of production functions that are often analyzed in both macroeconomics and microeconomics, namely homogeneous and homothetic production functions. Various geometric properties of these production models were obtained in the last period of time by many geometers, but there are some non-homogeneous production functions, including the famous Spillman-Mitscherlich and transcendental production functions, which were not enough investigated from a differential geometric point of view.

We recall that the Spillman-Mitscherlich production function is defined by

$$(8) \quad f(x_1, \dots, x_n) = A \cdot [1 - \exp(-a_1 x_1)] \cdot \dots \cdot [1 - \exp(-a_n x_n)],$$

where A, a_1, \dots, a_n are positive constants. On the other hand, the transcendental production function is given by

$$(9) \quad f(x_1, \dots, x_n) = A \cdot x_1^{a_1} \exp(b_1 x_1) \cdot \dots \cdot x_n^{a_n} \exp(b_n x_n),$$

where A is a positive constant and $a_1, b_1, \dots, a_n, b_n$ are real constants (usually taken from the closed interval $[0, 1]$), such that

$$a_i^2 + b_i^2 \neq 0, \quad i = 1, \dots, n.$$

We remark that the Spillman-Mitscherlich and transcendental production functions belong to a more general class of production functions, namely of the form

$$(10) \quad f(x_1, \dots, x_n) = \prod_{i=1}^n g_i(x_i),$$

where g_1, \dots, g_n are continuous positive real functions with nowhere zero first derivatives. We say that a production function of the form (10) is a *product* production function. In particular, we note that the generalized Cobb-Douglas production function is also a particular type of product production function. We also remark that the product production function can be also generalized as follows. A production function is said to be *quasi-product*, if the function has the form

$$(11) \quad f(x_1, \dots, x_n) = F \left(\prod_{i=1}^n g_i(x_i) \right),$$

where F, g_1, \dots, g_n are continuous positive functions with nowhere zero first derivatives on their domain of definition. We note that quasi-product production functions include the generalized CES production functions. We also remark that Y. Fu and W.G. Wang obtained in [15] the following classification of quasi-product productions, provided its corresponding graph hypersurfaces are flat spaces.

Theorem 3.1. [15] *Let f be a quasi-product production function given by (11). If the corresponding production hypersurface is flat, then, up to translations, f is given by one of the following functions:*

- (a) $f(x_1, \dots, x_n) = F \left(\exp \left(\sum_{i=1}^n c_i x_i \right) \right)$, where $c_i \in \mathbb{R} - \{0\}$, $i \in \{1, \dots, n\}$;
- (b) $f(x_1, \dots, x_n) = C_1 \ln(g_1(x_1)) + \sum_{i=2}^n C_i x_i$, where f_1 satisfies $g_1 g_1'' \neq g_1'^2$ and $C_i \in \mathbb{R} - \{0\}$, $i \in \{1, \dots, n\}$;
- (c) $f(x_1, \dots, x_n) = A \sqrt{x_1 \cdot \dots \cdot x_n}$, where A is a positive constant.

We recall that, if f is a production function with n inputs x_1, x_2, \dots, x_n , $n \geq 2$, then the *elasticity of production* with respect to a certain factor of production x_i is defined as

$$(12) \quad E_{x_i} = \frac{x_i}{f} f_{x_i}$$

and the *marginal rate of technical substitution* of input x_j for input x_i is given by

$$(13) \quad \text{MRS}_{ij} = \frac{f_{x_j}}{f_{x_i}}.$$

A production function is said to satisfy *the proportional marginal rate of substitution property* if and only if

$$(14) \quad \text{MRS}_{ij} = \frac{x_i}{x_j},$$

for all $1 \leq i \neq j \leq n$.

On the other hand, it is well known that the most common quantitative indices of production factor substitutability are forms of the elasticity of substitution [18]. We recall that there are two concepts of elasticity of substitution: Hicks elasticity of substitution and Allen elasticity of substitution. The function $H_{ij} : \mathbb{R}_+^n \rightarrow \mathbb{R}$ defined by

$$(15) \quad H_{ij}(x_1, \dots, x_n) = \frac{\frac{1}{x_i f_{x_i}} + \frac{1}{x_j f_{x_j}}}{-\frac{f_{x_i x_i}}{f_{x_i}^2} + \frac{2f_{x_i x_j}}{f_{x_i} f_{x_j}} - \frac{f_{x_j x_j}}{f_{x_j}^2}},$$

for all $(x_1, \dots, x_n) \in \mathbb{R}_+^n$, is called the *Hicks elasticity of substitution* of the i th production factor with respect to the j th production factor, where $i, j \in \{1, \dots, n\}$, $i \neq j$.

Moreover, the function $A_{ij} : \mathbb{R}_+^n \rightarrow \mathbb{R}$ defined as

$$A_{ij}(x_1, \dots, x_n) = -\frac{x_1 f_{x_1} + \dots + x_n f_{x_n}}{x_i x_j} \frac{\Delta_{ij}}{\Delta},$$

for all $(x_1, \dots, x_n) \in \mathbb{R}_+^n$, where Δ is the determinant of the bordered matrix

$$\begin{pmatrix} 0 & f_{x_1} & \dots & f_{x_n} \\ f_{x_1} & f_{x_1 x_1} & \dots & f_{x_1 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_n} & f_{x_n x_1} & \dots & f_{x_n x_n} \end{pmatrix}$$

and Δ_{ij} is the co-factor of the element f_{ij} in the determinant Δ ($\Delta \neq 0$ is assumed), is said to be the *Allen elasticity of substitution* of the i th production factor with respect to the j th production factor, where $i, j \in \{1, \dots, n\}$, $i \neq j$. Moreover, Δ is usually called the *Allen determinant*.

We note that in case of two inputs, Hicks elasticity of substitution and Allen elasticity of substitution coincide. So $H_{ij} = A_{ij}$ and in this case the indicator is simply called the *elasticity of substitution* between the two factors of production. We note that the elasticity of substitution was originally introduced by J.R. Hicks [16] in case of two inputs for the purpose of analyzing changes in the income shares of labor and capital. A twice differentiable production function f with nowhere zero

first partial derivatives is said to satisfy the CES (constant elasticity of substitution) property if there is a nonzero real constant σ such that

$$(16) \quad H_{ij}(x_1, \dots, x_n) = \sigma,$$

for $(x_1, \dots, x_n) \in \mathbb{R}_+^n$ and $1 \leq i \neq j \leq n$.

We remark that B.-Y. Chen [8] has completely classified homogeneous production functions which satisfy the CES property, generalizing to an arbitrary number of inputs an earlier result of L. Losonczy [17] for two inputs. Moreover, the classification has been later extended to the classes of quasi-sum and homothetic production functions in [10, 11]. We note that quasi-sum production functions are of great interest because they appear as solutions to the general bisymmetry equation, being related to the problem of consistent aggregation [2]. On the other hand, A. Mihai, M.E. Aydin and M. Ergüt classified quasi-sum and quasi-product production functions by their Allen determinants [3, 5].

4. SOME CLASSIFICATION RESULTS

Theorem 4.1. *Let f be a quasi-product production function given by (11), where the functions F, g_1, \dots, g_n are twice differentiable. Then:*

- i. *The elasticity of production is a constant k_i with respect to a certain factor of production x_i if and only if f reduces to*

$$(17) \quad f(x_1, \dots, x_n) = A \cdot x_i^{k_i} \cdot \prod_{j \neq i} g_j^k(x_j),$$

where A is a positive constant and k is a nonzero real constant.

- ii. *The elasticity of production is constant with respect to all factors of production if and only if f reduces to the generalized Cobb-Douglas production function given by (1).*
- iii. *The production function satisfies the proportional marginal rate of substitution property if and only if it reduces to the homothetic generalized Cobb-Douglas production function given by*

$$(18) \quad f(x_1, \dots, x_n) = F \left(A \cdot \prod_{i=1}^n x_i^k \right),$$

where A is a positive constant and k is a nonzero real constant.

- iv. *If the production function satisfies the proportional marginal rate of substitution property, then:*
 - iv₁. *The production hypersurface cannot be minimal.*
 - iv₂. *The production hypersurface has vanishing sectional curvature if and only if, up to a suitable translation, f reduces to the following generalized Cobb-Douglas production function:*

$$(19) \quad f(x_1, \dots, x_n) = A \cdot \sqrt{x_1 \dots x_n},$$

where A is a positive constant.

- v. *The production hypersurface has vanishing Gauss-Kronecker curvature if and only if, up to a suitable translation, f reduces to the one of the following:*
 - (a) *a generalized Cobb-Douglas production function with constant return to scale;*

- (b) $f(x_1, \dots, x_n) = A \cdot \ln \left[\exp(A_1 x_1) \cdot \prod_{j=2}^n g_j(x_j) \right]$, where A, A_1 are nonzero real constants;
- (c) $f(x_1, \dots, x_n) = F \left(A \cdot \exp(A_1 x_1 + A_2 x_2) \cdot \prod_{j=3}^n g_j(x_j) \right)$, where A is a positive constant and A_1, A_2 are nonzero real constants;
- (d) an Armington aggregator with constant return to scale, given by

$$f(x_1, \dots, x_n) = \left(\sum_{i=1}^n C_i x_i^{\frac{A-1}{A}} \right)^{\frac{A}{A-1}},$$

where A is a nonzero real constant, $A \neq 1$, and C_1, \dots, C_n are nonzero real constants;

- (e) $f(x_1, \dots, x_n) = A \cdot \ln \left(\sum_{i=1}^n B_i \exp(A_i x_i) \right)$, where A, A_i, B_i are nonzero real constants for $i = 1, \dots, n$.
- vi. The production function satisfies the constant elasticity of substitution property if and only if, up to a suitable translation, f reduces to the one of the following:
- (a) a homothetic generalized Cobb-Douglas production function given by

$$f(x_1, \dots, x_n) = F \left(A \cdot \prod_{i=1}^n x_i^{k_i} \right),$$

where A is a positive constant and k_1, \dots, k_n are nonzero real constants.

- (b) $f(x_1, \dots, x_n) = F \left(A \cdot \prod_{i=1}^n \exp \left(A_i x_i^{\frac{\sigma-1}{\sigma}} \right) \right)$, where A is a positive constant and A_1, \dots, A_n, σ are nonzero real constants, $\sigma \neq 1$;
- (c) a two-input production function given by

$$f(x_1, x_2) = F \left(A \cdot \left(\frac{x_1^{\frac{\sigma-1}{\sigma}} + A_1}{x_2^{\frac{\sigma-1}{\sigma}} + A_2} \right)^{\frac{\sigma}{k}} \right),$$

where A, A_1, A_2, k, σ are nonzero real constants, $\sigma \neq 1$;

- (d) a two-input production function given by

$$f(x_1, x_2) = F \left(A \cdot \left(\frac{\ln(A_1 x_1)}{\ln(A_2 x_2)} \right)^{\frac{1}{k}} \right),$$

where A, k are nonzero real constants and A_1, A_2 are positive constants.

Proof. Let f be a quasi-product production function given by (11). In what follows we will use the notation $u = g_1(x_1) \cdot \dots \cdot g_n(x_n)$. Then we have

$$(20) \quad f_{x_i} = u F' \frac{g'_i}{g_i},$$

where F' denotes the derivative with respect to the variable u and $g'_i = \frac{dg_i}{dx_i}$, for $i = 1, \dots, n$.

From (20) we derive

$$(21) \quad f_{x_i x_i} = u^2 F'' \left(\frac{g'_i}{g_i} \right)^2 + u F' \frac{g''_i}{g_i}, \quad i = 1, \dots, n$$

and

$$(22) \quad f_{x_i x_j} = u(u F'' + F') \frac{g'_i g'_j}{g_i g_j}, \quad i \neq j.$$

i. We first prove the left-to-right implication. If the elasticity of production is a constant k_i with respect to a certain factor of production x_i , then from (12) we obtain

$$(23) \quad f_{x_i} = k_i \frac{f}{x_i}.$$

Using now (11) and (20) in (23) we get

$$(24) \quad u \frac{F'}{F} = \frac{k_i}{x_i} \cdot \frac{g_i}{g'_i}.$$

By taking the partial derivative of (24) with respect to x_j , $j \neq i$, we obtain

$$u \frac{g'_j}{g_j} \frac{(F' + u F'') F - u F'^2}{F^2} = 0.$$

Hence, because $u > 0$ and $g'_j \neq 0$, we deduce that

$$(25) \quad \left(\frac{u F'}{F} \right)' = 0.$$

We obtain now easily that the solution of (25) is

$$(26) \quad F(u) = C \cdot u^k,$$

for some constants $C > 0$ and $k \in \mathbb{R} - \{0\}$. From (26) and (24) we derive

$$\frac{g'_i}{g_i} = \frac{k_i}{k x_i}$$

with solution

$$(27) \quad g_i(x_i) = B \cdot x_i^{\frac{k_i}{k}}.$$

where B is a positive constant.

Finally, combining (11), (26) and (27) we get a function of the form (17), where $A = C \cdot B^k$.

The converse is easy to verify by direct computation.

ii. The assertion is an immediate consequence of i.

iii. Let us assume first that f satisfies the proportional marginal rate of substitution property. Then taking account of (13), (14) and (20) we obtain

$$x_j \frac{g'_j}{g_j} = x_i \frac{g'_i}{g_i}, \quad \forall i \neq j.$$

Therefore we derive that there exists a nonzero real number k such that

$$x_i \frac{g'_i}{g_i} = k, \quad i = 1, \dots, n,$$

and we get

$$(28) \quad g_i(x_i) = A_i x_i^k, \quad i = 1, \dots, n,$$

for some positive constants A_1, \dots, A_n .

From (11) and (28) we derive that

$$f(x_1, \dots, x_n) = F \left(A \prod_{i=1}^n x_i^k \right),$$

where $A = \prod_{i=1}^n A_i$ and the conclusion follows.

The converse can be verified easily by direct computation.

iv. We assume that the production function given by (11) satisfies the proportional marginal rate of substitution property. Then we deduce from (18) that, denoting $G = F \circ g$, where $g(t) = A \cdot t^k$, the function f takes the form

$$(29) \quad f(x_1, \dots, x_n) = G \left(\prod_{i=1}^n x_i \right).$$

Therefore we have

$$(30) \quad f_{x_i} = \frac{uG'}{x_i},$$

$$(31) \quad f_{x_i x_i} = \frac{u^2 G''}{x_i^2}$$

and

$$(32) \quad f_{x_i x_j} = \frac{u(G' + uG'')}{x_i x_j},$$

where $u = \prod_{i=1}^n x_i$,

If the corresponding production hypersurface of f is minimal, then we have $H = 0$ and from (5) we derive

$$(33) \quad \sum_{i=1}^n f_{x_i x_i} + \sum_{i \neq j} (f_{x_i}^2 f_{x_j x_j} - f_{x_i} f_{x_j} f_{x_i x_j}) = 0.$$

By introducing (30), (31) and (32) in (33) we obtain:

$$(34) \quad u^2 G'' \sum_{i=1}^n \frac{1}{x_i^2} - u^3 G'^3 \sum_{i \neq j} \frac{1}{x_i^2 x_j^2} = 0.$$

But the only solution of the equation (34) is $G(u) = \text{constant}$, which is a contradiction. Hence the assertion (iv₁) follows.

Let us assume now that the production hypersurface of f has $K_{ij} = 0$. Then from (6) we derive

$$(35) \quad f_{x_i x_i} f_{x_j x_j} - f_{x_i x_j}^2 = 0.$$

By introducing (30), (31) and (32) in (35), since $G' \neq 0$, we obtain

$$\frac{G''}{G'} = -\frac{1}{2u}.$$

Therefore we get immediately

$$(36) \quad G(u) = A\sqrt{u} + B$$

for some constants A, B , with $A \neq 0$.

Combining now (29) and (36), after a suitable translation, we conclude that the function f reduces to the Cobb-Douglas production function given by (19). Since the converse can be easily verified by direct computation, the assertion (iv₂) follows.

v. We first suppose that the production hypersurface has null Gauss-Kronecker curvature. Then we derive from (4) that

$$(37) \quad \det(f_{x_i x_j}) = 0.$$

On the other hand, making use (21) and (22), we obtain that the determinant of the Hessian matrix of a composite function of the form (11) is given by [4]

$$(38) \quad \det(f_{x_i x_j}) = (uF')^n \left[\prod_{j=1}^n \left(\frac{g'_j}{g_j} \right)' + \left(1 + u \frac{F''}{F'} \right) \sum_{j=1}^n \left(\left(\frac{g'_j}{g_j} \right)^2 \cdot \prod_{i \neq j} \left(\frac{g'_i}{g_i} \right)' \right) \right].$$

We divide now the proof of the theorem into two main cases: (A) and (B).

Case (A): $\frac{g'_1}{g_1}, \dots, \frac{g'_n}{g_n}$ are nonconstant. Then, from (37) and (38) we derive

$$(39) \quad 1 + \left(1 + u \frac{F''}{F'} \right) \sum_{j=1}^n \frac{\left(\frac{g'_j}{g_j} \right)^2}{\left(\frac{g'_j}{g_j} \right)'} = 0.$$

We remark that for the above equation to have solution, it is necessary to have

$$1 + u \frac{F''}{F'} \neq 0 \text{ and } \sum_{j=1}^n \frac{\left(\frac{g'_j}{g_j} \right)^2}{\left(\frac{g'_j}{g_j} \right)'} \neq 0. \text{ In this case, (39) reduces to}$$

$$(40) \quad \sum_{j=1}^n \frac{\left(\frac{g'_j}{g_j} \right)^2}{\left(\frac{g'_j}{g_j} \right)'} = - \frac{F'}{F' + uF''}.$$

By taking the partial derivative of (40) with respect to x_i and dividing both sides of the derived expression by $\frac{g'_i}{g_i}$, we obtain

$$(41) \quad 2 - \frac{\frac{g'_i}{g_i} \cdot \left(\frac{g'_i}{g_i} \right)''}{\left[\left(\frac{g'_i}{g_i} \right)' \right]^2} = u \cdot \frac{F' F'' + u[F' F''' - (F'')^2]}{(F' + uF'')^2}.$$

Therefore, after taking the partial derivative of (41) with respect to x_j , with $j \neq i$, and simplifying the derived expression by $\left(u \cdot \frac{g'_j}{g_j} \right)$ we get

$$\begin{aligned} & \frac{F' F'' + u[F' F''' - (F'')^2]}{(F' + uF'')^2} + u \cdot \frac{2F' F''' + u(F' F^{iv} - F'' F''')}{(F' + uF'')^2} \\ & - 2u \cdot \frac{[F' F'' + u(F' F''' - (F'')^2)](2F'' + uF''')}{(F' + uF'')^3} = 0. \end{aligned}$$

We remark now that making the substitution

$$G = \frac{F'F'' + u[F'F''' - (F'')^2]}{(F' + uF'')^2},$$

the above equation reduces to

$$G + uG' = 0,$$

with solution $G(u) = \frac{A}{u}$, where A is a real constant. Hence we derive that

$$\frac{F'F'' + u[F'F''' - (F'')^2]}{(F' + uF'')^2} = \frac{A}{u},$$

which is equivalent to

$$(42) \quad \left(-\frac{F'}{F' + uF''} \right)' = \frac{A}{u}.$$

From (42) we find

$$(43) \quad -\frac{F'}{F' + uF''} = A \ln u + B,$$

for some real constants A, B .

We divide now the proof of case (A) into several cases.

Case (A.1) $A = 0$; In this case it follows that $B \neq 0$ and (43) implies that

$$(44) \quad 1 + u \frac{F''}{F'} = -\frac{1}{B}.$$

On the other hand, from (40) we deduce that

$$(45) \quad \sum_{i=1}^n \frac{\left(\frac{g'_i}{g_i} \right)^2}{\left(\frac{g'_i}{g_i} \right)^r} = B$$

for any nonzero real constant B .

Solving (45) we find

$$(46) \quad g_i(x_i) = A_i(x_i + B_i)^{-k_i},$$

for some constants A_i, B_i, k_i , with $A_i \neq 0$ and $k_i \neq 0$, $i = 1, \dots, n$, such that

$$\sum_{i=1}^n k_i = B.$$

On the other hand, (44) reduces to

$$(47) \quad \frac{F''}{F'} = -\frac{B+1}{Bu}.$$

Case (A.1.1): $B = -1$. Then from (47) we derive that $F(u) = Cu + D$, for some real constants C, D , with $C \neq 0$, and combining with (11) and (46), we conclude that, after a suitable translation, the function f reduces to a generalized Cobb-Douglas production function with constant return to scale. Hence we obtain the case (a) of the theorem.

Case (A.1.2): $B \neq -1$. Then we obtain easily that the solution of (47) is

$$(48) \quad F(u) = C \cdot u^{-\frac{1}{B}} + D,$$

for some real constants C, D , with $C \neq 0$.

Combining now (11), (46) and (48), after a suitable translation, we conclude that f reduces to the following function

$$f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^n x_i^{\frac{k_i}{B}},$$

where A is a positive constant. But it is obvious that $\sum_{i=1}^n \frac{k_i}{B} = 1$ and therefore we deduce that the above function is a generalized Cobb-Douglas production function with constant return to scale. Hence we obtain again the case (a) of the theorem. *Case (A.2) $A \neq 0$* ; In this case it follows that it is necessary to have $A \ln u + B \neq 0$ and we derive from (43) that

$$\frac{F''}{F'} = -\frac{1}{u(A \ln u + B)} - \frac{1}{u}.$$

Hence we obtain

$$(49) \quad F'(u) = \frac{C}{u(A \ln u + B)^{\frac{1}{A}}},$$

where C is a nonzero real constant. Now, from (49), we get that

$$(50) \quad F(u) = D(\ln u + E)^{-\frac{1}{A}+1} + F,$$

where D is a nonzero real constant and E, F are real constant, provided that $A \neq 1$. On the other hand, if $A = 1$, then we obtain from (49) that

$$(51) \quad F(u) = C \ln(\ln u + B) + D,$$

where D is a real constant.

But we can easily see now that (41) implies

$$(52) \quad 2 - \frac{\frac{g'_i}{g_i} \cdot \left(\frac{g'_i}{g_i}\right)''}{\left[\left(\frac{g'_i}{g_i}\right)'\right]^2} = A,$$

for $i = 1, \dots, n$.

Case (A.2.1) $A=2$; In this case we obtain from (50)

$$(53) \quad F(u) = D\sqrt{\ln u + E} + F,$$

and from (52) it follows that

$$\left(\frac{g'_i}{g_i}\right)'' = 0.$$

Hence we derive

$$(54) \quad g_i(x_i) = \exp(a_i x_i^2 + b_i x_i + c_i), \quad i = 1, \dots, n,$$

where a_i, b_i, c_i are real constants. Because $g'_i \neq 0$ on \mathbb{R}_+ , it follows that the constants a_i and b_i must satisfy the following conditions: $a_i b_i \geq 0$ and $a_i^2 + b_i^2 \neq 0$, for $i = 1, \dots, n$. Combining now (11), (53) and (54) we deduce that f takes the form

$$(55) \quad f(x_1, \dots, x_n) = D \cdot \sqrt{\sum_{i=1}^n A_i (x_i + B_i)^2 + E} + F,$$

for some constants A_i, B_i, E, F with $A_i \neq 0$. Now, making use of Lemma 2.1(i), it is direct to verify that the production hypersurface associated with the production function given by (55) has vanishing Gauss-Kronecker curvature if and only if $E = 0$. Hence, after a suitable translation we obtain the case (d) of the theorem with $A = 2$. *Case (A.2.2) $A \neq 2$* ; We deduce from (52) that

$$(56) \quad \frac{\frac{g'_i}{g_i} \cdot \left(\frac{g'_i}{g_i}\right)''}{\left[\left(\frac{g'_i}{g_i}\right)'\right]^2} = 2 - A,$$

Denoting $h_i = \frac{g'_i}{g_i}$, we obtain from (56)

$$(57) \quad \left(\frac{h_i}{h'_i}\right)' = A - 1$$

Case (A.2.2.i.) $A = 1$; Then from (57) we conclude that $\frac{h_i}{h'_i} = \bar{A}_i$, where \bar{A}_i is a nonzero real constant ($i = 1, \dots, n$) and we deduce

$$\frac{g'_i}{g_i} = D_i \exp(A_i x_i), \quad i = 1, \dots, n,$$

where D_i is a real constant and $A_i = (\bar{A}_i)^{-1}$ for $i = 1, \dots, n$. Now we can derive immediately that

$$(58) \quad g_i(x_i) = C_i \exp(B_i \exp(A_i x_i)), \quad i = 1, \dots, n,$$

where B_i is a nonzero constant and C_i is a positive constant.

Combining now (11), (51) and (58) we deduce that f takes the form

$$(59) \quad f(x_1, \dots, x_n) = C \ln \left(\sum_{i=1}^n B_i \exp(A_i x_i) + B \right) + D$$

for some nonzero constants C, A_i, B_i , $i = 1, \dots, n$, and real constants B, D . Now, making use of Lemma 2.1(i), it follows by direct computation that the production hypersurface associated with the production function given by (59) has vanishing Gauss-Kronecker curvature if and only if $B = 0$. Hence, after a suitable translation, we obtain the case (e) of the theorem.

Case (A.2.2.ii.) $A \neq 1$; Then from (57) we derive that $\frac{h'_i}{h_i} = \frac{1}{(A-1)x_i + A_i}$, where A_i is a real constant ($i = 1, \dots, n$) and we obtain

$$(60) \quad \frac{g'_i}{g_i} = B_i [(A-1)x_i + A_i]^{\frac{1}{A-1}}, \quad i = 1, \dots, n,$$

where B_i is a nonzero real constant, $i = 1, \dots, n$.

From (60) we obtain

$$(61) \quad g_i(x_i) = C_i \exp \left(\frac{B_i}{A} [(A-1)x_i + A_i]^{\frac{A}{A-1}} \right), \quad i = 1, \dots, n,$$

where C_i is a positive constant, $i = 1, \dots, n$.

Combining (11), (50) and (61) we deduce that f takes the form

$$(62) \quad f(x_1, \dots, x_n) = D \left(\sum_{i=1}^n \frac{B_i}{A} [(A-1)x_i + A_i]^{\frac{A}{A-1}} + B \right)^{\frac{A-1}{A}} + F$$

for some constants D, A_i, B_i, B, F , with $D > 0$ and $B_i \neq 0$. Now, using Lemma 2.1(i), we can easily verify that the production hypersurface associated with the production function given by (62) has vanishing Gauss-Kronecker curvature if and only if $B = 0$. Hence, after a suitable translation we obtain the case (d) of the theorem.

Case (B): at least one of $\frac{g'_1}{g_1}, \dots, \frac{g'_n}{g_n}$ is constant. Without loss of generality, we may assume that $\frac{g'_1}{g_1} = A_1$, where A_1 is a nonzero real constant. Then we derive that

$$(63) \quad g_1(x_1) = B_1 \exp(A_1 x_1),$$

where B_1 is a positive constant. Then (37) and (38) imply

$$(64) \quad \left(1 + u \frac{F''}{F'}\right) \cdot \prod_{i=2}^n \left(\frac{g'_i}{g_i}\right)' = 0.$$

From (64) we derive that either $1 + u \frac{F''}{F'} = 0$ or $\prod_{i=2}^n \left(\frac{g'_i}{g_i}\right)' = 0$. But in the first case we derive immediately that

$$(65) \quad F(u) = A \ln u + B,$$

where A is a nonzero real constant and B is a real constant. Hence, from (11), (63) and (65) we deduce that, after a suitable translation, we obtain the case (b) of the theorem.

On the other hand, in the second case we may assume without loss of generality that $\left(\frac{g'_2}{g_2}\right)' = 0$. Hence we get

$$(66) \quad g_2(x_2) = B_2 \exp(A_2 x_2),$$

where A_2 is a nonzero real constant and B_2 is a positive constant.

Combining now (11), (63) and (66), we obtain the case (c) of the theorem.

Conversely, we can verify by direct computation that all of the production hypersurfaces defined by the production functions in cases (a)-(e) of the theorem have vanishing Gauss-Kronecker curvature.

vi. We first assume that the production function satisfies the constant elasticity of substitution property. Then using (15), (20), (21) and (22) in (16) we obtain

$$(67) \quad \begin{aligned} & \sigma x_i x_j u^3 (F')^3 \left[2 \left(\frac{g'_i}{g_i}\right)^2 \left(\frac{g'_j}{g_j}\right)^2 - \frac{g''_i}{g_i} \left(\frac{g'_j}{g_j}\right)^2 - \frac{g''_j}{g_j} \left(\frac{g'_i}{g_i}\right)^2 \right] = \\ & = u^3 (F')^3 \frac{g'_i}{g_i} \frac{g'_j}{g_j} \left(x_i \frac{g'_i}{g_i} + x_j \frac{g'_j}{g_j} \right) \end{aligned}$$

and taking into account that $x_i > 0$, $i = 1, \dots, n$, and F, g_1, \dots, g_n are positive functions with nowhere zero first derivatives, we get from (67):

$$(68) \quad \sigma \left[2 - \frac{g_i g''_i}{(g'_i)^2} - \frac{g_j g''_j}{(g'_j)^2} \right] = \frac{1}{x_i} \frac{g_i}{g'_i} + \frac{1}{x_j} \frac{g_j}{g'_j}.$$

Now, it is easy to see that (68) can be written as

$$(69) \quad \frac{1}{x_i} \frac{g_i}{g'_i} - \sigma \left(\frac{g_i}{g'_i}\right)' + \frac{1}{x_j} \frac{g_j}{g'_j} - \sigma \left(\frac{g_j}{g'_j}\right)' = 0,$$

for $1 \leq i \neq j \leq n$.

Next, we can divide the proof into two separate cases.

Case A. $n \geq 3$. Then it is obvious that (69) implies

$$(70) \quad \frac{1}{x_i} \frac{g_i}{g_i'} - \sigma \left(\frac{g_i}{g_i'} \right)' = 0, \quad i = 1, \dots, n,$$

and we derive easily that the solution of (70) is

$$(71) \quad g_i(x_i) = \begin{cases} B_i \exp \left(C_i x_i^{\frac{\sigma-1}{\sigma}} \right), & \sigma \neq 1 \\ B_i x_i^{C_i}, & \sigma = 1 \end{cases},$$

for some positive constants B_i and nonzero real constants C_i , $i = 1, \dots, n$. Combining now (11) and (71) we get cases (a) and (b) of the theorem.

Case B. $n = 2$. Then it follows from (69) that

$$(72) \quad \begin{cases} \frac{1}{x_1} \frac{g_1}{g_1'} - \sigma \left(\frac{g_1}{g_1'} \right)' = k \\ \frac{1}{x_2} \frac{g_2}{g_2'} - \sigma \left(\frac{g_2}{g_2'} \right)' = -k \end{cases},$$

for some constant k . We remark now that, if $k = 0$, then we obtain immediately the cases (a) and (b) of the theorem with $n = 2$. Next we consider that $k \neq 0$. Then solving (72), we derive

$$(73) \quad g_1(x_1) = \begin{cases} B_1 \left(\frac{k}{\sigma-1} x_1^{\frac{\sigma-1}{\sigma}} + C_1 \right)^{\frac{\sigma}{k}}, & \sigma \neq 1 \\ B_1 (k \ln x_1 + C_1)^{\frac{1}{k}}, & \sigma = 1 \end{cases}$$

and

$$(74) \quad g_2(x_2) = \begin{cases} B_2 \left(\frac{k}{\sigma-1} x_2^{\frac{\sigma-1}{\sigma}} + C_2 \right)^{-\frac{\sigma}{k}}, & \sigma \neq 1 \\ B_2 (k \ln x_2 + C_2)^{-\frac{1}{k}}, & \sigma = 1 \end{cases}$$

for some constants B_1, B_2, C_1, C_2 . Combining now (11), (73) and (74) we get cases (c) and (d) of the theorem.

The converse follows easily by direct computation. \square

From Theorem 4.1 we derive immediately the following result for quasi-product production models with two inputs.

Corollary 4.2. *Let f be a quasi-product production function with two-inputs given by*

$$f(K, L) = F(g(K) \cdot h(L))$$

where K is the capital and L is the labor. Then:

- i. *f has a constant elasticity of capital k if and only if f reduces to*

$$f(K, L) = A \cdot K^k \cdot h(L),$$

where A is a positive constant.

- ii. *f has a constant elasticity of labor ℓ if and only if f reduces to*

$$f(K, L) = A \cdot L^\ell \cdot g(K),$$

where A is a positive constant.

- iii. f has constant elasticities with respect to both labor and capital if and only if f reduces to the Cobb-Douglas production function given by

$$f(K, L) = AK^k L^\ell,$$

where A is a positive constant.

- iv. f satisfies the proportional marginal rate of substitution property between capital and labor if and only if f is a homothetic Cobb-Douglas production function given by

$$f(K, L) = F(AK^k L^k),$$

where A is a positive constant and k is a nonzero real constant.

- v. If the production function satisfies the proportional marginal rate of substitution property, then the associated production surface cannot be minimal.
- vi. The production surface is developable if and only if, up to a suitable translation, f reduces to the one of the following:
- (a) a two-input Cobb-Douglas production function with constant return to scale;
 - (b) $f(K, L) = A \cdot \ln[\exp(A_1 K)h(L)]$, where A, A_1 are nonzero real constants;
 - (c) $f(K, L) = F(A \cdot \exp(A_1 K + A_2 L))$, where A is a positive constant and A_1, A_2 are nonzero real constants;
 - (d) a two-input Armington aggregator with constant return to scale given by

$$f(K, L) = \left(C_1 K^{\frac{A}{A-1}} + C_2 L^{\frac{A}{A-1}} \right)^{\frac{A-1}{A}},$$

where A is a nonzero real constant, $A \neq 1$, and C_1, C_2 are nonzero real constants;

- (e) $f(K, L) = A \cdot \ln(B_1 \exp(A_1 K) + B_2 \exp(A_2 L))$, where A, A_1, B_1, A_2, B_2 are nonzero real constants.
- vii. The production function satisfies the constant elasticity of substitution property if and only if, up to a suitable translation, f reduces to the one of the following:
- (a) a homothetic Cobb-Douglas production function given by

$$f(K, L) = F(AK^{k_1} L^{k_2}),$$

where A is a positive constant and k_1, k_2 are nonzero real constants.

- (b) $f(K, L) = F\left(A \cdot \exp\left(A_1 K^{\frac{\sigma-1}{\sigma}}\right) \exp\left(A_2 L^{\frac{\sigma-1}{\sigma}}\right)\right)$, where A is a positive constant and σ, A_1, \dots, A_n are nonzero real constants, $\sigma \neq 1$;
- (c) $f(K, L) = F\left(A \cdot \left(\frac{K^{\frac{\sigma-1}{\sigma}} + A_1}{L^{\frac{\sigma-1}{\sigma}} + A_2}\right)^{\frac{\sigma}{k}}\right)$, where A, A_1, A_2, k, σ are nonzero real constants, $\sigma \neq 1$;
- (d) $f(K, L) = F\left(A \cdot \left(\frac{\ln(A_1 K)}{\ln(A_2 L)}\right)^{\frac{1}{k}}\right)$, where A, k are nonzero real constants and A_1, A_2 are positive constants.

From Theorems 3.1 and 4.1 we obtain easily the following result for product production models.

Corollary 4.3. *Let f be a product production function given by (10), where the functions g_1, \dots, g_n are twice differentiable. Then:*

- i. The elasticity of production is a constant k_i with respect to a certain factor of production x_i if and only if f reduces to

$$f(x_1, \dots, x_n) = A \cdot x_i^{k_i} \cdot \prod_{j \neq i} g_j(x_j),$$

where A is a positive constant and k is a nonzero real constant.

- ii. The elasticity of production is constant with respect to all factors of production if and only if f reduces to the generalized Cobb-Douglas production function given by (1).
 iii. The production function satisfies the proportional marginal rate of substitution property if and only if it reduces to the generalized Cobb-Douglas production function given by

$$f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^n x_i^k,$$

where A is a positive constant and k is a nonzero real constant.

- iv. If the production function satisfies the proportional marginal rate of substitution property, then:
 iv₁. The production hypersurface cannot be minimal.
 iv₂. The production hypersurface has vanishing sectional curvature if and only if, up to a suitable translation, f reduces to the generalized Cobb-Douglas production function given by (19).
 v. The production hypersurface has vanishing Gauss-Kronecker curvature if and only if, up to a suitable translation, f reduces to the one of the following:
 (a) a generalized Cobb-Douglas production function with constant return to scale;
 (b) $f(x_1, \dots, x_n) = A \cdot \exp(A_1 x_1 + A_2 x_2) \cdot \prod_{j=3}^n g_j(x_j)$, where A is a positive constant and A_1, A_2 are nonzero real constants.
 vi. The production function satisfies the constant elasticity of substitution property if and only if, up to a suitable translation, f reduces to the one of the following:
 (a) a generalized Cobb-Douglas production function given by (1).
 (b) $f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^n \exp\left(A_i x_i^{\frac{\sigma-1}{\sigma}}\right)$, $\sigma \neq 1$, where A is a positive constant and A_1, \dots, A_n, σ are nonzero real constants, $\sigma \neq 1$;
 (c) a two-input production function given by

$$f(x_1, x_2) = A \cdot \left(\frac{x_1^{\frac{\sigma-1}{\sigma}} + A_1}{x_2^{\frac{\sigma-1}{\sigma}} + A_2} \right)^{\frac{\sigma}{k}},$$

where A, A_1, A_2, k, σ are nonzero real constants, $\sigma \neq 1$;

- (d) a two-input production function given by

$$f(x_1, x_2) = A \cdot \left(\frac{\ln(A_1 x_1)}{\ln(A_2 x_2)} \right)^{\frac{1}{k}},$$

where A, k are nonzero real constants and A_1, A_2 are positive constants.

viii. The production hypersurface is flat if and only if, up to a suitable translation, f reduces to the one of the following:

- (a) $f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^n \exp(C_i x_i)$, where A is a positive constant and C_1, \dots, C_n are nonzero real constants;
- (b) A generalized Cobb-Douglas production function given by

$$f(x_1, \dots, x_n) = A \sqrt{x_1 \cdot \dots \cdot x_n},$$

where A is a positive constant.

From Corollary 4.3 we obtain the following results concerning Spillman-Mitscherlich and transcendental production functions.

Corollary 4.4. *Let f be a Spillman-Mitscherlich production function given by (8). Then:*

- i. The elasticity of production cannot be constant with respect to the factor of production x_i , $i = 1, \dots, n$.
- ii. f does not satisfy the proportional marginal rate of substitution property.
- iii. f does not satisfy the constant elasticity of substitution property.
- iv. The production hypersurface of f has non-vanishing Gauss-Kronecker curvature.
- v. The production hypersurfaces of f is non-flat.

Corollary 4.5. *Let f be a transcendental production function given by (9). Then:*

- i. The elasticity of production is constant with respect to a certain factor of production x_i if and only if $b_i = 0$.
- ii. The elasticity of production is constant with respect to all factors of production if and only if $b_1 = \dots = b_n = 0$.
- iii. f satisfies the proportional marginal rate of substitution property if and only if $a_1 = \dots = a_n \neq 0$ and $b_1 = \dots = b_n = 0$. Moreover, in this case, the production hypersurface of f cannot be minimal, but it has vanishing sectional curvature if and only if $a_1 = \dots = a_n = \frac{1}{2}$.
- iv. The production hypersurface of f has vanishing Gauss-Kronecker curvature if and only if one of the following situations occurs:
 - (a) $a_1 + \dots + a_n = 1$ and $b_1 = \dots = b_n = 0$;
 - (b) There are two different indices $i, j \in \{1, \dots, n\}$ such that $a_i = a_j = 0$.
- v. f satisfies the constant elasticity of substitution property if and only if $b_1 = \dots = b_n = 0$;
- vi. The production hypersurfaces of f is flat if and only if one of the following situations occurs:
 - (a) $a_1 = \dots = a_n = 0$;
 - (b) $a_1 = \dots = a_n = \frac{1}{2}$ and $b_1 = \dots = b_n = 0$.

ACKNOWLEDGMENTS

This Project was funded by the National Plan for Science, Technology and Innovation (MAARIFAH), King Abdulaziz City for Science and Technology, Kingdom of Saudi Arabia, Award Number (15-MAT4092-02).

REFERENCES

- [1] K. Abe, H. Okamoto, M. Tawada, *A note on the production possibility frontier with pure public intermediate goods*, Can. J. Econ. **19**(2) (1986), 351-356.
- [2] J. Aczél, G. Maksa, *Solution of the rectangular $m \times n$ generalized bisymmetry equation and of the problem of consistent aggregation*, J. Math. Anal. Appl. **203** (1996), 104-126.
- [3] M.E. Aydin, M. Ergüt, *Composite functions with Allen determinants and their applications to production models in economics*, Tamkang J. Math. **45**(4) (2014), 427-435.
- [4] M.E. Aydin, M. Ergüt, *Hessian determinants of composite functions with applications for production functions in economics*, Kragujevac J. Math. **38**(2) (2014), 259-268.
- [5] M.E. Aydin, A. Mihai, *Classification of quasi-sum production functions with Allen determinants*, Filomat **29**(6) (2015), 1351-1359.
- [6] B.-Y. Chen, *Geometry of submanifolds*, M. Dekker, New York, 1973.
- [7] B.-Y. Chen, *Pseudo-Riemannian geometry, δ -invariants and applications*, World Scientific, Hackensack, NJ, 2011.
- [8] B.-Y. Chen, *Classification of h -homogeneous production functions with constant elasticity of substitution*, Tamkang J. Math. **43**(2) (2012), 321-328.
- [9] B.-Y. Chen, *On some geometric properties of quasi-sum production models*, J. Math. Anal. Appl. **392**(2) (2012), 192-199.
- [10] B.-Y. Chen, *Geometry of quasi-sum production functions with constant elasticity of substitution property*, J. Adv. Math. Stud. **5**(2) (2012), 90-97.
- [11] B.-Y. Chen, *Classification of homothetic functions with constant elasticity of substitution and its geometric applications*, Int. Electron. J. Geom. **5**(2) (2012), 67-78.
- [12] B.-Y. Chen, *Solutions to homogeneous Monge-Ampère equations of homothetic functions and their applications to production models in economics*, J. Math. Anal. Appl. **411** (2014), 223-229.
- [13] B.-Y. Chen, G.E. Vilcu, *Geometric classifications of homogeneous production functions*, Appl. Math. Comput. **225** (2013), 345-351.
- [14] C.W. Cobb, P.H. Douglas, *A theory of production*, Am. Econ. Rev. **18** (1928), 139-165.
- [15] Y. Fu, W.G. Wang, *Geometric characterizations of quasi-product production models in economics*, Filomat (2016), to be published.
- [16] J.R. Hicks, *Theory of wages*, London, Macmillan, 1932.
- [17] L. Losonczi, *Production functions having the CES property*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **26**(1) (2010), 113-125.
- [18] D. McFadden, *Constant Elasticity of Substitution Production Functions*, The Rev. Econ. Stud. **30**(2) (1963), 73-83.
- [19] T. Inoue, *On the shape of the production possibility frontier with more commodities than primary factors*, Int. Econ. Rev. **25**(2) (1984), 409-424.
- [20] S.K. Mishra, *A Brief History of Production Functions*, IUP J. Manag. Econom. **8**(4) (2010), 6-34.
- [21] H. Uzawa, *Production Functions with Constant Elasticities of Substitution*, The Rev. Econ. Stud. **29**(4) (1962), 291-299.
- [22] G.E. Vilcu, *A geometric perspective on the generalized Cobb-Douglas production functions*, Appl. Math. Lett. **24**(5) (2011), 777-783.
- [23] A.D. Vilcu, G.E. Vilcu, *On some geometric properties of the generalized CES production functions*, Appl. Math. Comput. **218**(1) (2011), 124-129.
- [24] A.D. Vilcu, G.E. Vilcu, *Some characterizations of the quasi-sum production models with proportional marginal rate of substitution*, C. R. Math. Acad. Sci. Paris, **353** (2015), 1129-1133.
- [25] X. Wang, *A geometric characterization of homogeneous production models in economics*, Filomat (2016), to be published.
- [26] X. Wang, Y. Fu, *Some characterizations of the Cobb-Douglas and CES production functions in microeconomics*, Abstr. Appl. Anal. **2013** (2013), Article ID 761832, 6 pages.

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